Q. How can you tell if the series $\sum_{n=0}^{\infty} a_n$ converges or diverges?

① geometric, integrable, telescoping

② compare $\sum a_n$ to something in ①

Ex 1 Does $\sum_{k=2}^{\infty} \frac{1}{2k-1}$ converge or diverge?

(a) Converges

(b) Diverges

$$\frac{1}{2k-1} \geq \frac{1}{2k}$$

$$\downarrow$$

$$\sum_{k=2}^{\infty} \frac{1}{2k-1} \geq \sum_{k=2}^{\infty} \frac{1}{2k} = \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{k} = \infty$$

diverge to ∞

Ex 2 Does $\sum_{n=1}^{\infty} \frac{1}{ne^n}$ converge or diverge?

(a) Converges

(b) Diverges

$$\frac{1}{ne^n} \leq \frac{1}{e^n} = \left(\frac{1}{e}\right)^n$$

$$\downarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{ne^n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{e}\left(\frac{1}{1-\frac{1}{e}}\right)$$

converges

Since $\frac{1}{nc^n} > 0$, S_k is monotone $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{ne^n}$ converges by MCT

Ex 3 Does $\sum_{n=1}^{\infty} -\frac{1}{n}$ converge or diverge?

$$-\sum_{n=1}^{\infty} \frac{1}{n} = -\infty$$

(a) Converges

(b) Diverges

Note: $-\frac{1}{n} \leq \frac{1}{n^2} \longrightarrow \sum_{n=1}^{\infty} -\frac{1}{n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

But $\sum_{n=1}^{\infty} -\frac{1}{n}$ diverges!

Moral: Being bounded above by a convergent series is not enough to conclude convergence

Thm: (The Direct Comparison Test) (want to know about / already know about)

(a) Suppose there is an index n_0 such that $0 \leq a_n \leq b_n$ for $n \geqslant n_0$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(b) Suppose there is an index n_0 such that $b_n \leq a_n$ for $n \geqslant n_0$. If $\sum_{n=1}^{\infty} b_n$ diverges to ∞, then $\sum_{n=1}^{\infty} a_n$ also diverges to ∞.

Ex 4 Decide whether $\sum_{n=1}^{\infty} \frac{1}{n^5 + \sqrt{n}}$ converges or diverges.

(a) Converges

(b) Diverges

$$0 \leq \frac{1}{n^5+\sqrt{n}} \leq \frac{1}{n^5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5+\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^5}$$ ← converges (by p-test with $p=5$)

By D.C.T. $\sum_{n=1}^{\infty} \frac{1}{n^5+\sqrt{n}}$ converges

Q. Could we use a different comparison?

Note: $0 \le \frac{1}{n^5+\sqrt{n}} \le \frac{1}{\sqrt{n}}$

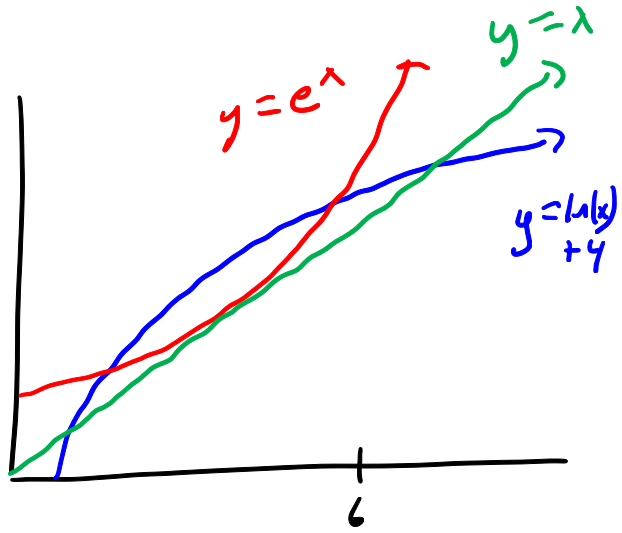
$\to \sum_{n=1}^{\infty} \frac{1}{n^5+\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

↑ diverges to ∞ by p-test with $p = \frac{1}{2}$

$\Rightarrow$ No conclusion

Ex 5 Decide whether $\sum_{n=1}^{\infty} \frac{1}{\ln(n)+4}$ converges or diverges.

(a) Converges
(b) Diverges



$\frac{1}{\ln(x)+4} \ge \frac{1}{e^x}$ (eventually)

$\sum_{n=1}^{\infty} \frac{1}{\ln(n)+4} \ge \sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e}\left(\frac{1}{1-\frac{1}{e}}\right)$

$\to$ no conclusion

$\frac{1}{\ln(x)+4} \ge \frac{1}{x}$ for $x > 6$

$\to \sum_{n=1}^{\infty} \frac{1}{\ln(n)+4} \ge \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ By D.C.T. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞

Q. Could we use a different comparison?

$$0 \le \frac{1}{\ln(n)+4} \le \frac{1}{4}$$

$$\to \sum_{n=1}^{\infty} \frac{1}{\ln(n)+4} \le \sum_{n=1}^{\infty} \frac{1}{4} = \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\cdots = \infty$$

$\Rightarrow$ No conclusion

Ex 6 What about $\sum_{n=1}^{\infty} \frac{5n}{5n^3-2}$?

(a) Converges

(b) Diverges

For large n: $\frac{5n}{5n^3-2} \approx \frac{5n}{5n^3} = \frac{1}{n^2}$

But $\frac{5n}{5n^3-2} \ge \frac{5n}{5n^3} = \frac{1}{n^2}$

so $\sum_{n=1}^{\infty} \frac{5n}{5n^3-2} \ge \sum_{n=1}^{\infty} \frac{1}{n^2}$ ← converges by p-test ($p=2$)

and being larger than something that converges is not helpful

Thm: (Limit Comparison Test) Suppose $a_n \geqslant 0$ and $b_n \geqslant 0$ for all $n \geqslant 1$, and suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ (where L is a stand-in for either a real number or ∞).

(a) If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same behavior. (i.e. both converge or both diverge)

(b) If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(c) If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges to ∞, then $\sum_{n=1}^{\infty} a_n$ diverges to ∞

Ex 7 Using $a_n = \frac{5n}{5n^3 - 2}$ and $b_n = \frac{1}{n^2}$ for all $n \geqslant 1$, decide whether $\sum_{n=1}^{\infty} \frac{5n}{5n^3 - 2}$ converges or diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{5n}{5n^3-2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2 \cdot 5n}{5n^3-2} = \lim_{n\to\infty} \frac{5n^3}{5n^3-2} = \lim_{n\to\infty} \frac{15n^2}{15n^2} = 1 \quad \left(\text{L'H}\frac{\infty}{\infty}\right)$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same behavior

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test ($p = 2$), $\sum_{n=1}^{\infty} \frac{5n}{5n^3-2}$ also converges

Ex 8 Decide whether $\sum_{n=1}^{\infty} \frac{8^n}{3+4^n}$ converges or diverges.

(a) Converges

(b) Diverges

Divergence test: $\lim_{n\to\infty} \frac{8^n}{3+4^n} \overset{\frac{\infty}{\infty}}{=} \lim_{n\to\infty} \frac{8^n \cdot \ln(8)}{4^n \cdot \ln(4)} = \lim_{n\to\infty} 2^n \cdot \frac{\ln(8)}{\ln(4)}$

$= \infty \Rightarrow \sum_{n=1}^{\infty} \frac{8^n}{3+4^n}$ diverges

LCT $a_n = \frac{8^n}{3+4^n} \approx 2^n = b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{8^n}{3+4^n}}{2^n} = \lim_{n\to\infty} \frac{8^n}{2^n(3+4^n)} = \lim_{n\to\infty} \frac{4^n}{3+4^n} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{4^n \ln(4)}{4^n \ln(4)} = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{8^n}{3+4^n} \text{ and } \sum_{n=1}^{\infty} 2^n \text{ have the same behavior}$$

Since $\sum_{n=1}^{\infty} 2^n$ diverges, so does $\sum_{n=1}^{\infty} \frac{8^n}{3+4^n}$

Ex 9 Decide (by any means) whether the series $\sum_{n=1}^{\infty} \frac{n^4+1}{e^n}$ converges or not.

(a) Converges

(b) Diverges

Divergence test: $\lim_{n\to\infty} \frac{n^4+1}{e^n} = \lim_{n\to\infty} \frac{4n^3}{e^n}$

$\frac{\infty}{\infty}$

$= \cdots = \lim_{n\to\infty} \frac{24}{e^n} = 0$

→ No conclusion

D.C.T. $\frac{n^4+1}{e^n} \geq \frac{1}{e^n}$ ← geometric

$\sum_{n=1}^{\infty} \frac{n^4+1}{e^n} \geq \sum_{n=1}^{\infty} \frac{1}{e^n}$ ← converges

→ No conclusion

L.C.T. $a_n = \frac{n^4+1}{e^n}$ $b_n = \frac{1}{e^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^4+1}{e^n}}{\left(\frac{1}{e^n}\right)} = \lim_{n\to\infty} \frac{e^n(n^4+1)}{e^n}$

$= \infty$

$\sum a_n \geq \sum b_n$ ← converge

⇒ No conclusion

L.C.T. $a_n = \frac{n^4+1}{e^n}$ $b_n = \frac{1}{2^n}$

↑ goes to 0 less quickly than $\frac{1}{e^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n^4+1}{e^n}\right)}{\left(\frac{1}{2^n}\right)}$$

$$= \lim_{n\to\infty} \frac{n^4+1}{e^n\left(\frac{1}{2^n}\right)}$$

$\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{n^4+1}{\left(\frac{e}{2}\right)^n}$$

$\frac{e}{2} > 1$

$$= \lim_{n\to\infty} \frac{4n^3}{\left(\frac{e}{2}\right)^n \cdot \ln\left(\frac{e}{2}\right)}$$

$$= \lim_{n\to\infty} \frac{12n^2}{\left(\frac{e}{2}\right)^n \cdot \ln\left(\frac{e}{2}\right)^2}$$

$$= \lim_{n\to\infty} \frac{24n}{\left(\frac{e}{2}\right)^n \cdot \ln\left(\frac{e}{2}\right)^3}$$

$$= \lim_{n\to\infty} \frac{24}{\left(\frac{e}{2}\right)^n \cdot \ln\left(\frac{e}{2}\right)^4} = 0$$

a_n is "smaller" than b_n (eventually), but $\sum_{n=1}^{\infty} b_n$ converges

By the L.C.T. $\sum_{n=1}^{\infty} a_n$ also converges.